9 Electrooptic Modulation of Laser Beams

9.0 INTRODUCTION

In Chapter 1 we treated the propagation of electromagnetic waves in anisotropic crystal media. It was shown how the properties of the propagating wave can be determined from the index ellipsoid surface.

In this chapter we consider the problem of propagation of optical radiation in crystals in the presence of an applied electric field. We find that in certain types of crystals it is possible to effect a change in the index of refraction that is proportional to the field. This is the linear electrooptic effect. It affords a convenient and widely used means of controlling the intensity or phase of the propagating radiation. This modulation is used in an ever expanding number of applications including: the impression of information onto optical beams, Q-switching of lasers (Section 6.9) for generation of giant optical pulses, mode locking, and optical beam deflection. Some of these applications will be discussed further in this chapter. Modulation and deflection of laser beams by acoustic beams are considered in Chapter 12.

9.1 ELECTROOPTIC EFFECT

In Chapter 1 we found that, given a direction in a crystal, in general two possible linearly polarized modes exist: the so-called rays of propagation. Each mode possesses a unique direction of polarization (that is, direction of **D**) and a corresponding index of refraction (that is, a velocity of propagation). The mutually orthogonal polarization directions and the indices of the two rays are found most easily by using the index ellipsoid

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1 \tag{9.1-1}$$

where the directions x, y, and z are the principal dielectric axes—that is, the directions in the crystal along which $\mathbf{D}$ and $\mathbf{E}$ are parallel. The existence of two rays (one "ordinary"; the other "extraordinary") with different indices of refraction is called *birefringence.*

The linear electrooptic effect is the change in the indices of the ordinary and extraordinary rays that is caused by and is proportional to an applied electric field. This effect exists only in crystals that do not possess inversion symmetry.[1] This statement can be justified as follows: Assume that in a crystal possessing an inversion symmetry, the application of an electric field E along some direction causes a change $\Delta n_1 = sE$ in the index, where s is a constant characterizing the linear electrooptic effect. If the direction of the field is reversed, the change in the index is given by $\Delta n_2 = s(-E)$, but because of the inversion symmetry the two directions are physically equivalent, so $\Delta n_1 = \Delta n_2$. This requires that $s = -s$, which is possible only for $s = 0$, so no linear electrooptic effect can exist. The division of all crystal classes into those that do and those that do not possess an inversion symmetry is an elementary consideration in crystallography and this information is widely tabulated [1].

Since the propagation characteristics in crystals are fully described by means of the index ellipsoid (9.1-1), the effect of an electric field on the propagation is expressed most conveniently by giving the changes in the constants $1/n_x^2$, $1/n_y^2$, $1/n_z^2$ of the index ellipsoid.

Following convention [1–2], we take the equation of the index ellipsoid in the presence of an electric field as

$$\left(\frac{1}{n^2}\right)_1 x^2 + \left(\frac{1}{n^2}\right)_2 y^2 + \left(\frac{1}{n^2}\right)_3 z^2 + 2\left(\frac{1}{n^2}\right)_4 yz + 2\left(\frac{1}{n^2}\right)_5 xz + 2\left(\frac{1}{n^2}\right)_6 xy = 1 \tag{9.1-2}$$

If we choose x, y, and z to be parallel to the principal dielectric axes of the crystal, then with zero applied field, Equation (9.1-2) must reduce to (9.1-1); therefore,

[1] If a crystal contains points (one in each unit cell) such that inversion (replacing each atom at $\mathbf{r}$ by one at $-\mathbf{r}$, with $\mathbf{r}$ being the position vector relative to the point) about any one of these points leaves the crystal structure invariant, the crystal is said to possess inversion symmetry.

$$\left(\frac{1}{n^2}\right)_1\bigg|_{E=0} = \frac{1}{n_x^2} \qquad \left(\frac{1}{n^2}\right)_2\bigg|_{E=0} = \frac{1}{n_y^2}$$
$$\left(\frac{1}{n^2}\right)_3\bigg|_{E=0} = \frac{1}{n_z^2} \qquad \left(\frac{1}{n^2}\right)_4\bigg|_{E=0} = \left(\frac{1}{n^2}\right)_5\bigg|_{E=0} = \left(\frac{1}{n^2}\right)_6\bigg|_{E=0} = 0$$

The linear change in the coefficients

$$\left(\frac{1}{n^2}\right)_i \qquad i = 1, \ldots, 6$$

due to an arbitrary dc electric field $\mathbf{E}(E_x, E_y, E_z)$ is defined by

$$\Delta\left(\frac{1}{n^2}\right)_i = \sum_{j=1}^{3} r_{ij}E_j \tag{9.1-3}$$

where in the summation over j we use the convention $1 = x$, $2 = y$, $3 = z$. Equation (9.1-3) can be expressed in a matrix form as

$$\begin{vmatrix} \Delta\left(\frac{1}{n^2}\right)_1 \\ \Delta\left(\frac{1}{n^2}\right)_2 \\ \Delta\left(\frac{1}{n^3}\right)_3 \\ \Delta\left(\frac{1}{n^2}\right)_4 \\ \Delta\left(\frac{1}{n^2}\right)_5 \\ \Delta\left(\frac{1}{n^2}\right)_6 \end{vmatrix} = \begin{vmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \\ r_{41} & r_{42} & r_{43} \\ r_{51} & r_{52} & r_{53} \\ r_{61} & r_{62} & r_{63} \end{vmatrix} \begin{vmatrix} E_1 \\ E_2 \\ E_3 \end{vmatrix} \tag{9.1-4}$$

where, using the rules for matrix multiplication, we have, for example,

$$\Delta\left(\frac{1}{n^2}\right)_6 = r_{61}E_1 + r_{62}E_2 + r_{63}E_3$$

The 6×3 matrix with elements r_{ij} is called the electrooptic tensor. We have shown above that in crystals possessing an inversion symmetry (centrosymmetric), $r_{ij} = 0$. The form, but not the magnitude, of the tensor r_{ij} can be derived from symmetry considerations [1], which dictate which of the 18 r_{ij} coefficients are zero, as well as the relationships that exist between the remaining coefficients. In Table 9-1 we give the form of the electrooptic tensor for all the noncentrosymmetric crystal classes. The electrooptic coefficients of some crystals are given in Table 9-2.

Table 9-1 The Form of the Electrooptic Tensor for all Crystal Symmetry Classes

Symbols:

$\cdot$ zero element

$\bullet$ nonzero element

$\bullet\!\!-\!\!\bullet$ equal nonzero elements

$\bullet\!\!-\!\!\circ$ equal nonzero elements, but opposite in sign

The symbol at the upper left corner of each tensor is the conventional symmetry group designation.

Centrosymmetric—All elements zero

Triclinic

$$\begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix}$$

Monoclinic

2 (parallel to x_2)

$$\begin{pmatrix} \cdot & \bullet & \cdot \\ \cdot & \bullet & \cdot \\ \cdot & \bullet & \cdot \\ \bullet & \cdot & \bullet \\ \cdot & \bullet & \cdot \\ \bullet & \cdot & \bullet \end{pmatrix}$$

(parallel to x_3)

$$\begin{pmatrix} \cdot & \cdot & \bullet \\ \cdot & \cdot & \bullet \\ \cdot & \cdot & \bullet \\ \bullet & \bullet & \cdot \\ \bullet & \bullet & \cdot \\ \cdot & \cdot & \bullet \end{pmatrix}$$

m (perpendicular to x_2)

$$\begin{pmatrix} \bullet & \cdot & \bullet \\ \bullet & \cdot & \bullet \\ \bullet & \cdot & \bullet \\ \cdot & \bullet & \cdot \\ \bullet & \cdot & \bullet \\ \cdot & \bullet & \cdot \end{pmatrix}$$

(perpendicular to x_3)

$$\begin{pmatrix} \bullet & \bullet & \cdot \\ \bullet & \bullet & \cdot \\ \bullet & \bullet & \cdot \\ \cdot & \cdot & \bullet \\ \cdot & \cdot & \bullet \\ \bullet & \bullet & \cdot \end{pmatrix}$$

Orthorhombic

222

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \bullet & \cdot & \cdot \\ \cdot & \bullet & \cdot \\ \cdot & \cdot & \bullet \end{pmatrix}$$

*mm*2

$$\begin{pmatrix} \cdot & \cdot & \bullet \\ \cdot & \cdot & \bullet \\ \cdot & \cdot & \bullet \\ \cdot & \bullet & \cdot \\ \bullet & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

Table 9-1 *(continued)*

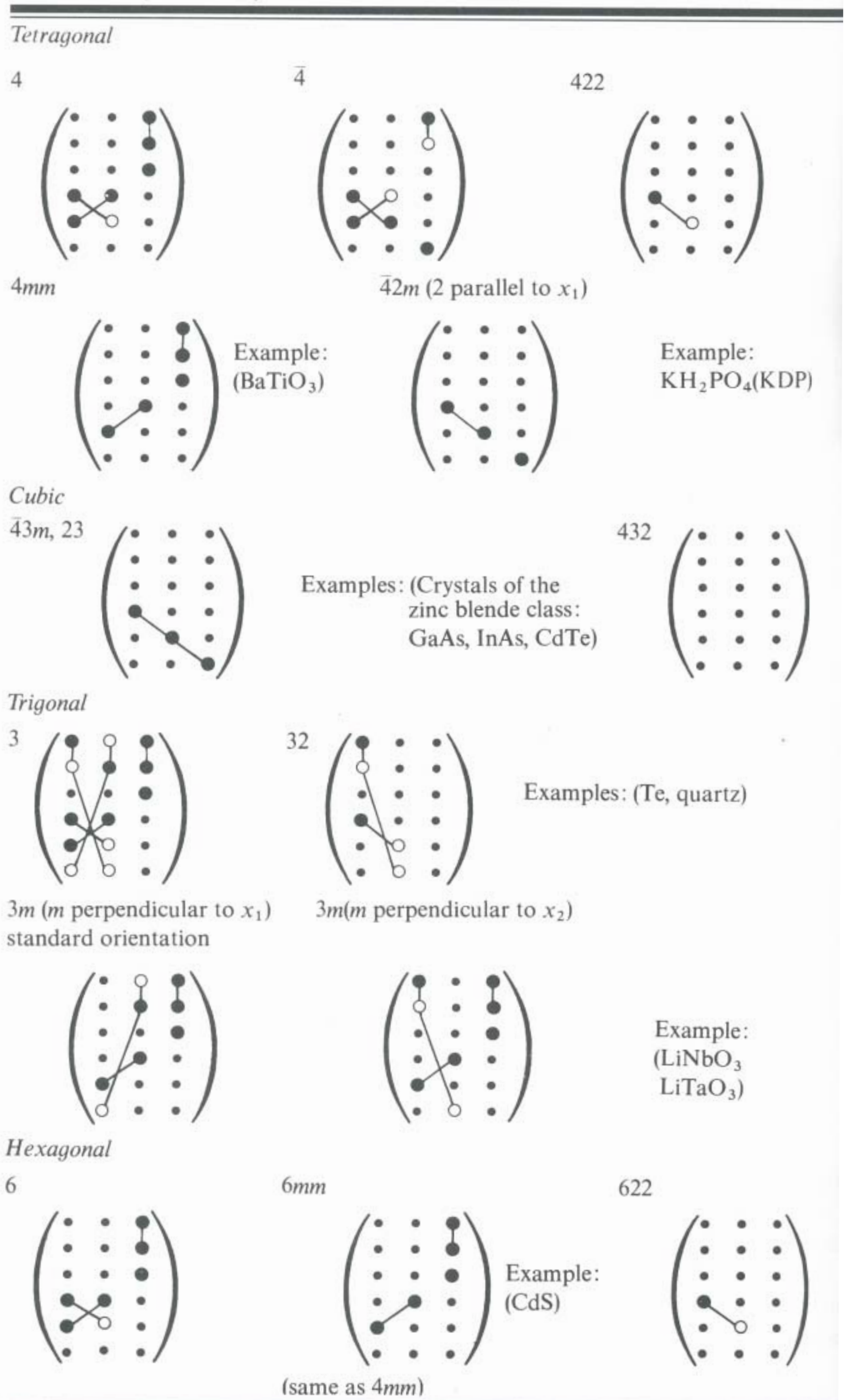


Table 9-1 *(continued)*

Tetragonal

4

$\bar{4}$

422

$4mm$

Example: ($BaTiO_3$)

$\bar{4}2m$ (2 parallel to x_1)

Example: KH_2PO_4(KDP)

Cubic

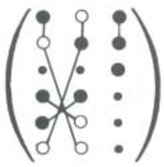
$\bar{4}3m$, 23

Examples: (Crystals of the zinc blende class: GaAs, InAs, CdTe)

432

Trigonal

3

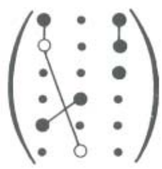


32

Examples: (Te, quartz)

$3m$ (m perpendicular to x_1) standard orientation

$3m$(m perpendicular to x_2)



Example: ($LiNbO_3$ $LiTaO_3$)

Hexagonal

6

$6mm$

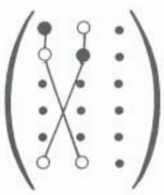
Example: (CdS)

(same as $4mm$)

622

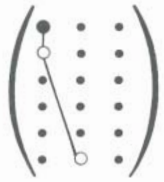
Table 9-1 *(continued)*

$\bar{6}$



$\bar{6}m2$ (m perpendicular to x_1 standard orientation)

(m perpendicular to x_2)



Example: The Electrooptic Effect in KH_2PO_4

Consider the specific example of a crystal of potassium dihydrogen phosphate (KH_2PO_4), also known as KDP. The crystal has a fourfold axis of symmetry,[2] which by strict convention is taken as the z (optic) axis, as well as two mutually orthogonal twofold axes of symmetry that lie in the plane normal to z. These are designated as the x and y axes. The symmetry group of this crystal is $\bar{4}2m$.[3] Using Table 9-1, we take the electrooptic tensor in the form of

$$r_{ij} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{41} & 0 & 0 \\ 0 & r_{41} & 0 \\ 0 & 0 & r_{63} \end{vmatrix} \tag{9.1-5}$$

so the only nonvanishing elements are $r_{41} = r_{52}$ and r_{63}. Using (9.1-2), (9.1-4), and (9.1-5), we obtain the equation of the index ellipsoid in the presence of a field $\mathbf{E}(E_x, E_y, E_z)$ as

$$\frac{x^2}{n_o^2} + \frac{y^2}{n_o^2} + \frac{z^2}{n_e^2} + 2r_{41}E_x yz + 2r_{41}E_y xz + 2r_{63}E_z xy = 1 \tag{9.1-6}$$

[2]That is, a rotation by $2\pi/4$ about this axis leaves the crystal structure invariant.

[3]The significance of the symmetry group symbols and a listing of most known crystals and their symmetry groups is to be found in any basic book on crystallography.

where the constants involved in the first three terms do not depend on the field and, since the crystal is uniaxial, are taken as $n_x = n_y = n_o$, $n_z = n_e$. We thus find that the application of an electric field causes the appearance of "mixed" terms in the equation of the index ellipsoid. These are the terms with xy, xz, and yz. This means that the major axes of the ellipsoid, with a field applied, are no longer parallel to the x, y, and z axes. It becomes necessary, then, to find the directions and magnitudes of the new axes, in the presence of $\mathbf{E}$, so that we may determine the effect of the field on the propagation. To be specific we choose the direction of the applied field parallel to the z axis, so (9.1-6) becomes

$$\frac{x^2 + y^2}{n_o^2} + \frac{z^2}{n_e^2} + 2r_{63}E_z xy = 1 \tag{9.1-7}$$

The problem is one of finding a new coordinate system—x', y', z'—in which the equation of the ellipsoid (9.1-7) contains no mixed terms; that is, it is of the form

$$\frac{x'^2}{n_{x'}^2} + \frac{y'^2}{n_{y'}^2} + \frac{z'^2}{n_{z'}^2} = 1 \tag{9.1-8}$$

x', y', and z' are then the directions of the major axes of the ellipsoid in the presence of an external field applied parallel to z. The length of the major axes of the ellipsoid is, according to (9.1-8), $2n_{x'}$, $2n_{y'}$, and $2n_{z'}$, and these will, in general, depend on the applied field.

In the case of (9.1-7) it is clear from inspection that in order to put the equation in a diagonal form we need to choose a coordinate system x', y', z' where z' is parallel to z, and because of the symmetry of (9.1-7) in x and y, x' and y' are related to x and y by a 45° rotation, as shown in Figure 9-1. The transformation relations from x, y to x', y' are thus

$$x = x' \cos 45° + y' \sin 45°$$
$$y = -x' \sin 45° + y' \cos 45°$$

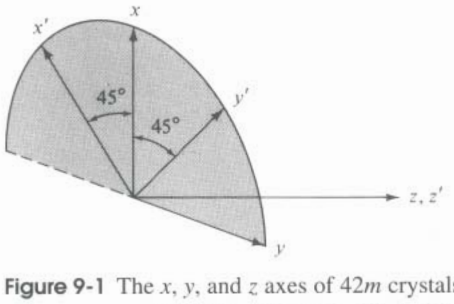


Figure 9-1 The x, y, and z axes of $\bar{4}2m$ crystals (such as KH_2PO_4) and the x', y', and z' axes, where z is the fourfold optic axis and x and y are the twofold axes of crystals with $\bar{4}2m$ symmetry.

which, upon substitution in (9.1-7), yield

$$\left(\frac{1}{n_o^2} - r_{63}E_z\right) x'^2 + \left(\frac{1}{n_o^2} + r_{63}E_z\right) y'^2 + \frac{z^2}{n_e^2} = 1 \tag{9.1-9}$$

Equation (9.1-9) shows that x', y', and z are indeed the principal axes of the ellipsoid when a field is applied along the z direction. According to (9.1-9), the length of the x' axis of the ellipsoid is $2n_{x'}$, where

$$\frac{1}{n_{x'}^2} = \frac{1}{n_o^2} - r_{63}E_z$$

which, assuming $r_{63}E_z \ll n_o^{-2}$ and using the differential relation $dn = -(n^3/2)\,d(1/n^2)$, gives for the change in $n_{x'}$, $dn_{x'} = -(n_o^3/2)r_{63}E_z$ so that

$$n_{x'} = n_o + \frac{n_o^3}{2} r_{63}E_z \tag{9.1-10}$$

and, similarly,

$$n_{y'} = n_o - \frac{n_o^3}{2} r_{63}E_z \tag{9.1-11}$$

$$n_z = n_e \tag{9.1-12}$$

The electrooptic effect in the practical imortant $\bar{4}3m$ crystal class (GaAs, InP, ZnS) is treated in detail in Appendix B.

The General Solution

We now consider the problem of optical propagation in a crystal in the presence of an external dc field along an arbitrary direction.

The index ellipsoid with the dc field on is given by (9.1-2), which we reexpress in the quadratic form

$$S_{ij}x_ix_j = 1 \tag{9.1-13}$$

so that $S_{11} = (1/n^2)_1$, $S_{32} = S_{23} = (1/n^2)_4$, and so on. We also use the convention of summation over repeated indices. Our problem consists of finding the directions and magnitudes of the principal axes of the ellipsoid (9.1-13).

Before proceeding we need remind ourselves of one basic result of vector calculus. If the vector from the origin to a point (x_1, x_2, x_3) on the ellipsoid (9.1-13) is denoted by $\mathbf{R}(x_1, x_2, x_3)$, then the vector $\mathbf{N}$ with components

$$N_i = S_{ij}x_j \tag{9.1-14}$$

is normal to the ellipsoid at $\mathbf{R}$.

We next apply the last result to determine the directions and magnitudes of the principal axes of the ellipsoid (9.1-13). Since the principal axes are normal to the

Table 9-2 Linear Electrooptic Coefficients of Some Commonly Used Crystals

Substance	Symmetry	Wavelength λ (μm)	Electrooptic Coefficients r_{lk} (10^{-12} m/V)	Index of Refraction n_i	n^3r (10^{-12} m/V)	Dielectric Constant* $\epsilon_i(\epsilon_0)$
CdTe (See App. B)	$\bar{4}3m$	1.0	$(T)\ r_{41} = 4.5$	$n = 2.84$	103	$(S)\ \epsilon = 9.4$
		3.39	$(T)\ r_{41} = 6.8$			
		10.6	$(T)\ r_{41} = 6.8$	$n = 2.60$	120	
		23.35	$(T)\ r_{41} = 5.47$	$n = 2.58$	94	
		27.95	$(T)\ r_{41} = 5.04$	$n = 2.53$	82	
GaAs (See App. B)	$\bar{4}3m$	0.9	$r_{41} = 1.1$	$n = 3.60$	51	$(S)\ \epsilon = 13.2$
		1.15	$(T)\ r_{41} = 1.43$	$n = 3.43$	58	$(T)\ \epsilon = 12.3$
		3.39	$(T)\ r_{41} = 1.24$	$n = 3.3$	45	
		10.6	$(T)\ r_{41} = 1.51$	$n = 3.3$	54	
GaP (See App. B)	$\bar{4}3m$	0.55–1.3	$(T)\ r_{41} = -1.0$	$n = 3.66–3.08$		$(S)\ \epsilon = 10$
		0.633	$(S)\ r_{41} = -0.97$	$n = 3.32$	35	
		1.15	$(S)\ r_{41} = -1.10$	$n = 3.10$	33	
		3.39	$(S)\ r_{41} = -0.97$	$n = 3.02$	27	
β-ZnS (sphalerite) (See App. B)	$\bar{4}3m$	0.4	$(T)\ r_{41} = 1.1$	$n = 2.52$	18	$(T)\ \epsilon = 16$
		0.5	$(T)\ r_{41} = 1.81$	$n = 2.42$		$(S)\ \epsilon = 12.5$
		0.6	$(T)\ r_{41} = 2.1$	$n = 2.36$		
		0.633	$(S)\ r_{41} = -1.6$	$n = 2.35$		
		3.39	$(S)\ r_{41} = -1.4$			
ZnSe (See App. B)	$\bar{4}3m$	0.548	$(T)\ r_{41} = 2.0$	$n = 2.66$		$(T)\ \epsilon = 9.1$
		0.633	$(S)\ r_{41} = 2.0$	$n = 2.60$	35	$(S)\ \epsilon = 9.1$
		10.6	$(T)\ r_{41} = 2.2$	$n = 2.39$		
ZnTe (See App. B)	$\bar{4}3m$	0.589	$(T)\ r_{41} = 4.51$	$n = 3.06$		$(T)\ \epsilon = 10.1$
		0.616	$(T)\ r_{41} = 4.27$	$n = 3.01$		$(S)\ \epsilon = 10.1$
		0.633	$(T)\ r_{41} = 4.04$	$n = 2.99$	108	
			$(S)\ r_{41} = 4.3$			
		0.690	$(T)\ r_{41} = 3.97$	$n = 2.93$		
		3.41	$(T)\ r_{41} = 4.2$	$n = 2.70$	83	
		10.6	$(T)\ r_{41} = 3.9$	$n = 2.70$	77	

$Bi_{12}SiO_{20}$	23	0.633	$r_{41} = 5.0$		$n = 2.54$	82	
CdSe	6 mm	3.39	$(S)\ r_{13} = 1.8$		$n_o = 2.452$		$(T)\epsilon_1 = 9.70$
							$(T)\ \epsilon_3 = 10.65$
			$(T)\ r_{33} = 4.3$		$n_e = 2.471$		$(S)\ \epsilon_1 = 9.33$
							$(S)\ \epsilon_3 = 10.20$
α-ZnS	6 mm	0.633	$(S)\ r_{13} = 0.9$		$n_o = 2.347$		$(T)\ \epsilon_1 = \epsilon_2 = 8.7$
(wurtzite)			$(S)\ r_{33} = 1.8$		$n_e = 2.360$		$(S)\ \epsilon_1 = 8.7$
$Pb_{0.814}La_{0.214}$-	$\infty\ m$	0.546	$n_e^3 r_{33} - n_o^3 r_{13} = 2320$		$n_o = 2.55$		
$(Ti_{0.6}Zr_{0.4})O_3$							
(PLZT)							
$LiIO_3$	6	0.633	$(S)\ r_{13} = 4.1$	$(S)\ r_{33} = 6.4$	$n_o = 1.8830$		
			$(S)\ r_{41} = 1.4$	$(S)\ r_{51} = 3.3$	$n_o = 1.7367$		
Ag_3AsS_3	$3m$	0.633	$(S)\ n_e^3 r_e = 70$		$n_o = 3.019$		
			$(S)\ n_o^3 r_{22} = 29$		$n_e = 2.739$		
$LiNbO_3$	$3m$	0.633	$(T_4)\ r_{13} = 9.6$	$(S)\ r_{13} = 8.6$	$n_o = {}_4 2.286$		$(T)\ \epsilon_1 = \epsilon_2 = 78$
$(T_c = 1230°C)$			$(T)\ r_{22} = 6.8$	$(S)\ r_{22} = 3.4$	$n_e = 2.200$		$(T)\ \epsilon_2 = 32$
			$(T)\ r_{33} = 30.9$	$(S)\ r_{33} = 30.8$			$(S)\ \epsilon_1 = \epsilon_2 = 43$
			$(T)\ r_{51} = 32.6$	$(S)\ r_{51} = 28$			$(S)\ \epsilon_3 = 28$
			$(T)\ r_c = 21.1$				
		1.15	$(T)\ r_{22} = 5.4$		$n_o = 2.229$		
			$(T)\ r_c = 19$		$n_e = 2.150$		
		3.39	$(T)\ r_{22} = 3.1$	$(S)\ r_{33} = 28$	$n_o = 2.136$		
			$(T)\ r_c = 18$	$(S)\ r_{22} = 3.1$	$n_e = 2.073$		
				$(S)\ r_{13} = 6.5$			
				$(S)\ r_{51} = 23$			

Table 9-2 *(continued)*

Substance	Symmetry	Wavelength λ (μm)	Electrooptic Coefficients r_{lk} (10^{-12} m/V)		Index of Refraction n_i	n^3r (10^{-12} m/V)	Dielectric Constant* $\epsilon_i(\epsilon_0)$
$LiTaO_3$	$3m$	0.633	$(T)\ r_{13} = 8.4$	$(S)\ r_{13} = 7.5$	$n_o = 2.176$		$(T)\ \epsilon_1 = \epsilon_2 = 51$
			$(T)\ r_{33} = 30.5$	$(S)\ r_{33} = 33$	$n_e = 2.180$		$(T)\ \epsilon_3 = 45$
			$(T)\ r_{22} = -0.2$	$(S)\ r_{51} = 20$			$(S)\ \epsilon_1 = \epsilon_2 = 41$
			$(T)\ r_c = 22$	$(S)\ r_{22} = 1$			$(S)\ \epsilon_3 = 43$
		3.39	$(S)\ r_{33} = 27$		$n_o = 2.060$		
			$(S)\ r_{13} = 4.5$		$n_e = 2.065$		
			$(S)\ r_{51} = 15$				
			$(S)\ r_{22} = 0.3$				
$AgGaS_2$	$\bar{4}2m$	0.633	$(T)\ r_{41} = 4.0$		$n_o = 2.553$		
			$(T)\ r_{63} = 3.0$		$n_e = 2.507$		
CsH_2AsO_4	$\bar{4}2m$	0.55	$(T)\ r_{41} = 14.8$		$n_o = 1.572$		
(CDA)			$(T)\ r_{63} = 18.2$		$n_e = 1.550$		
KH_2PO_4	$\bar{4}2m$	0.546	$(T)\ r_{41} = 8.77$		$n_o = 1.5115$		$(T)\ \epsilon_1 = \epsilon_2 = 42$
(KDP)			$(T)\ r_{63} = 10.3$		$n_e = 1.4698$		$(T)\ \epsilon_3 = 21$
		0.633	$(T)\ r_{41} = 8$		$n_o = 1.5074$		$(S)\ \epsilon_1 = \epsilon_2 = 44$
			$(T)\ r_{63} = 11$		$n_e = 1.4669$		$(S)\ \epsilon_3 = 21$
		3.39	$(T)\ r_{63} = 9.7$				
			$(T)\ n_o^3 r_{63} = 33$				
KD_2PO_4	$\bar{4}2m$	0.546	$(T)\ r_{63} = 26.8$		$n_o = 1.5079$		$(T)\ \epsilon_3 = 50$
(KD*P)			$(T)\ r_{41} = 8.8$		$n_e = 1.4683$		$(S)\ \epsilon_1 = \epsilon_2 = 58$
		0.633	$(T)\ r_{63} = 24.1$		$n_o = 1.502$		$(S)\ \epsilon_3 = 48$
					$n_e = 1.462$		
$(NH_4)H_2PO_4$	$\bar{4}2m$	0.546	$(T)\ r_{41} = 23.76$		$n_o = 1.5266$		$(T)\ \epsilon_1 = \epsilon_2 = 56$
(ADP)			$(T)\ r_{63} = 8.56$		$n_e = 1.4808$		$(T)\ \epsilon_3 = 15$
		0.633	$(T)\ r_{41} = 23.41$		$n_o = 1.5220$		$(S)\ \epsilon_1 = \epsilon_2 = 58$
			$(T)\ n_o^3 r_{63} = 27.6$		$n_e = 1.4773$		$(S)\ \epsilon_3 = 14$

$(NH_4)D_2PO_4$	$\bar{4}2m$	0.633	$(T)\ r_{41} = 40$		$n_o = 1.516$	
(AD*P)			$(T)\ r_{63} = 10$		$n_e = 1.475$	
$BaTiO_3$	4 mm	0.546	$(T)\ r_{51} = 1640$	$(S)\ r_{51} = 820$	$n_o = 2.437$	$(T)\ \epsilon_1 = \epsilon_2 = 3600$
$(T_c = 395$ K)			$(T)\ r_c = 108$	$(S)\ r_c = 23$	$n_e = 2.365$	$(T)\ \epsilon_3 = 135$
$(KTa_xNb_{1-x}O_3$		0.633	$(T)\ r_{51} = 8000(T_c - 28)$		$n_o = 2.318$	
(KTN),						
$x = 0.35$			$(T)\ r_c = 500(T_c - 28)$		$n_e = 2.277$	
$(T_c = 40$–$60°$C)			$(T)\ r_{51} = 3000(T_c - 16)$		$n_o = 2.318$	
			$(T)\ r_c = 700(T_c - 16)$		$n_e = 2.281$	
$Ba_{0.25}Sr_{0.75}Nb_2O_6$	4 mm	0.633	$(T)\ r_{13} = 67$	$(T)\ r_{51} = 42$	$n_o = 2.3117$	$\epsilon_3 = 3400$
$(T_c = 395$ K)			$(T)\ r_{33} = 1340$	$(S)\ r_c = 1090$	$n_e = 2.2987$	(15 MHz)
α-HIO_3	222	0.633	$(T)\ r_{41} = 6.6$	$(S)\ r_{41} = 2.3$	$n_1 = 1.8365$	
			$(T)\ r_{52} = 7.0$	$(S)\ r_{52} = 2.6$	$n_2 = 1.984$	
			$(T)\ r_{63} = 6.0$	$(S)\ r_{63} = 4.3$	$n_3 = 1.960$	
$KNbO_3$	2 mm	0.633	$(T)\ r_{13} = 28$	$(T)\ r_{23} = 1.3$ $(T)\ r_{33} = 64$	$n_1 = 2.280$	
			$(T)\ r_{42} = 380$	$(S)\ r_{42} = 270$	$n_2 = 2.329$	
			$(T)\ r_{51} = 105$		$n_3 = 2.169$	
KIO_3	1	0.500	$r_{62} = 90$		$n_1 = 1.700$	
					$n_2 = 1.828$ (5893 Å)	
					$n_3 = 1.832$	

*"(T)" = low frequency from dc through audio range; (S) = high frequency.

surface, we can determine their points of intersection (x_1, x_2, x_3) with the ellipsoid by requiring that at such points the radius vector be parallel to the normal, that is,

$$S_{ij}x_j = Sx_i \tag{9.1-15}$$

where S is a constant independent of i.

Writing out (9.1-15) in component form for $i = 1, 2, 3$ gives

$$\begin{aligned}(S_{11} - S)x_1 + S_{12}x_2 + S_{13}x_3 &= 0\\ S_{21}x_1 + (S_{22} - S)x_2 + S_{23}x_3 &= 0\\ S_{31}x_1 + S_{32}x_2 + (S_{33} - S)x_3 &= 0\end{aligned} \tag{9.1-16}$$

(9.1-16) constitutes a system of three homogeneous equations for the unknowns x_1, x_2, and x_3. The condition for a nontrivial solution is that the determinant of the coefficients vanishes, that is,

$$\det[S_{ij} - S\delta_{ij}] = 0 \tag{9.1-17}$$

This is a cubic equation in S. For real S_{ij}, which is the case with lossless crystals, the three roots S', S'', and S''' of (9.1-17) are real numbers. Having solved (9.1-17) we use the three roots, one at a time, in (9.1-16) to solve, to within a multiplicative constant, for the radius vector (x_1, x_2, x_3) to the point of intersection of the principal axis with the ellipsoid. The first vector, obtained by using S', is denoted by $\mathbf{X}'(x_1', x_2', x_3')$, the second by $\mathbf{X}''(x_1'', x_2'', x_3'')$, and the third, obtained from S''', is $\mathbf{X}'''(x_1''', x_2''', x_3''')$. Since the vectors satisfy (9.1-15), we have

$$S_{ij}x_j' = S'x_i' \tag{9.1-18}$$

with a similar relation applying to x_i'' and x_i'''.

It is an easy task to prove that the three principal axis vectors $\mathbf{X}'$, $\mathbf{X}''$, $\mathbf{X}'''$ are mutually orthogonal.

So far we have solved for the directions of the principal axes. Next we obtain their magnitudes. We multiply (9.1-18) by x_i'

$$S_{ij}x_i'x_j' = S'x_i'x_i' = S'|\mathbf{X}'|^2 \tag{9.1-19}$$

But the left side of (9.1-19) is, according to (9.1-13), equal to unity since the point (x_1', x_2', x_3') is on the ellipsoid (9.1-13). We can thus write

$$|\mathbf{X}'| = \frac{1}{\sqrt{S'}}$$

with similar results for $\mathbf{X}''$ and $\mathbf{X}'''$. The lengths of the principal axes of the index ellipsoid are thus $2(S')^{-1/2}$, $2(S'')^{-1/2}$, and $2(S''')^{-1/2}$. If we then express the equation of the index ellipsoid in terms of a Cartesian coordinate system whose axes are parallel to $\mathbf{X}'$, $\mathbf{X}''$, and $\mathbf{X}'''$, it becomes

$$S'x'^2 + S''y'^2 + S'''z'^2 = 1 \tag{9.1-20}$$

where the unit vectors $\mathbf{x}'$, $\mathbf{y}'$, and $\mathbf{z}'$ here are taken as parallel to $\mathbf{X}'$, $\mathbf{X}''$, and $\mathbf{X}'''$, respectively.

The bit of mathematics starting with (9.1-13) is referred to as the transformation of a quadratic form to a principal coordinate system. An equivalent description of this transformation is by the term matrix diagonalization. The original matrix being the ordered array of the coefficients S_{ij}

$$\mathbf{S} \equiv \begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{vmatrix} \tag{9.1-21}$$

The set of S', S'', and S''', which are the roots of (9.1-17), are the *eigenvalues* of the matrix $\mathbf{S}$, while the vectors $\mathbf{X}'$, $\mathbf{X}''$, and $\mathbf{X}'''$ are its eigenvectors. The term matrix diagonalization follows from the fact that if we express the quadric surface

$$S_{ij}x_ix_j = 1$$

whose coefficients form the matrix $\mathbf{S}$ of (9.1-21), in terms of a Cartesian coordinate system whose axes are $\mathbf{X}'$, $\mathbf{X}''$, and $\mathbf{X}'''$, it assumes the form (9.1-20) with the diagonal form of the matrix $\mathbf{S}$ as

$$\mathbf{S} = \begin{bmatrix} S' & 0 & 0 \\ 0 & S'' & 0 \\ 0 & 0 & S''' \end{bmatrix} \tag{9.1-22}$$

Example: Electrooptic Field in KH_2PO_4

To illustrate the method of matrix diagonalization, we use the example of KH_2PO_4(KDP) with a dc field along the crystal z axis, which was solved above in a somewhat less formal fashion.

The index ellipsoid is given by (9.1-7) as

$$\frac{x^2}{n_0^2} + \frac{y^2}{n_0^2} + \frac{z^2}{n_e^2} + 2r_{63}E_zxy = 1 \tag{9.1-23}$$

The S_{ij} matrix is thus

$$S_{ij} = \begin{bmatrix} \dfrac{1}{n_0^2} & r_{63}E_z & 0 \\ r_{63}E_z & \dfrac{1}{n_0^2} & 0 \\ 0 & 0 & \dfrac{1}{n_e^2} \end{bmatrix} \tag{9.1-24}$$

The eigenvalues are given according to (9.1-17) as the roots of the equation

$$\det \begin{vmatrix} \dfrac{1}{n_0^2} - S & r_{63}E_z & 0 \\ r_{63}E_z & \dfrac{1}{n_0^2} - S & 0 \\ 0 & 0 & \dfrac{1}{n_e^2} - S \end{vmatrix} \tag{9.1-25}$$

which upon evaluation is

$$\left(\frac{1}{n_e^2} - S\right)\left[\left(\frac{1}{n_0^2} - S\right)^2 - (r_{63}E_z)^2\right] = 0$$

The roots are

$$S' = \frac{1}{n_e^2}$$

$$S'' = \frac{1}{n_0^2} + r_{63}E_z$$

$$S''' = \frac{1}{n_0^2} - r_{63}E_z \tag{9.1-26}$$

in agreement with (9.1-9). These roots are used, one at a time, in the equation

$$S_{ij}x_j = Sx_i \qquad i = 1, 2, 3 \tag{9.1-27}$$

to obtain the eigenvectors. Starting with S' we have

$$\left(\frac{1}{n_0^2} - \frac{1}{n_e^2}\right) x_1' + r_{63}E_z x_2' = 0$$

$$r_{63}E_z x_1' + \left(\frac{1}{n_0^2} - \frac{1}{n_e^2}\right) x_2' = 0$$

$$\left(\frac{1}{n_e^2} - \frac{1}{n_e^2}\right) x_3' = 0 \tag{9.1-28}$$

The first two equations above are satisfied by $x_1' = 0$ and $x_2' = 0$, while the third is satisfied by any value of x_3'. The eigenvector $\mathbf{X}'$ corresponding to $S'(=1/n_e^2)$ is thus parallel to the z axis. In a like fashion we substitute the value of S'' into (9.1-27) and find that the corresponding eigenvector $\mathbf{X}''$ is parallel to the direction $\mathbf{x} + \mathbf{y}$ while using S''' shows that $\mathbf{X}'''$ is parallel to $\mathbf{x} - \mathbf{y}$. Referring to the last two eigenvector directions as x' and y', we can rewrite the equation of the index ellipsoid in the x', y', z (principal) coordinate system as

$$\left(\frac{1}{n_0^2} - r_{63}E_z\right) x'^2 + \left(\frac{1}{n_0^2} + r_{63}E_z\right) y'^2 + \frac{z^2}{n_e^2} = 1 \tag{9.1-29}$$

where the quantities in parentheses are the eigenvalues given by (9.1-26). Equation (9.1-29) is the same as (9.1-9).

9.2 ELECTROOPTIC RETARDATION

The index ellipsoid for KDP with **E** applied parallel to z is shown in Figure 9-2. If we consider propagation along the z direction, then according to the procedure described in Section 1.4 we need to determine the ellipse formed by the intersection of the plane $z = 0$ (in general, the plane that contains the origin and is normal to the propagation direction) and the ellipsoid. The equation of this ellipse is obtained from (9.1-9) by putting $z = 0$ and is

$$\left(\frac{1}{n_o^2} - r_{63}E_z\right) x'^2 + \left(\frac{1}{n_o^2} + r_{63}E_z\right) y'^2 = 1 \tag{9.2-1}$$

One quadrant of the ellipse is shown shaded in Figure 9-2, along with its minor and major axes, which in this case coincide with x' and y', respectively. It follows from Section 1.4 that the two allowed directions of polarization are x' and y' and that their indices of refraction are $n_{x'}$ and $n_{y'}$, which are given by (9.1-10) and (9.1-11).

We are now in a position to take up the concept of retardation. We consider an optical field that is incident normally on the $x'y'$ plane with its **E** vector along the x direction. We can resolve the optical field at $z = 0$ (input plane) into two mutually orthogonal components polarized along x' and y'. The x' component propagates as

$$e_{x'} = Ae^{i[\omega t - (\omega/c)n_{x'}z]}$$

which, using (9.1-10), becomes

$$e_{x'} = Ae^{i\{\omega t - (\omega/c)[n_o + (n_o^3/2)r_{63}E_z]z\}} \tag{9.2-2}$$

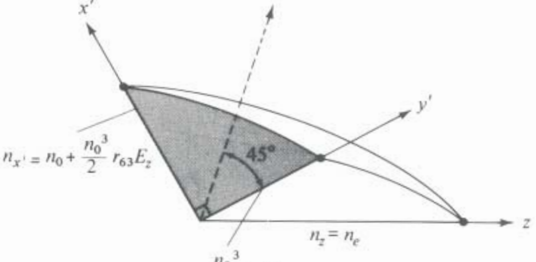


Figure 9-2 A section of the index ellipsoid of KDP, showing the principal dielectric axes x', y', and z due to an electric field applied along the z axis. The directions x' and y' are defined by Figure 9-1.